Demonstration by reductio ad impossibile in Posterior Analytics 1.26
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1. Aristotle’s thesis in Posterior Analytics 1.26
In Posterior Analytics 1.26, Aristotle argues that direct demonstrations are better than demonstrations by reductio ad impossibile:

T1 Since positive demonstration is better than privative demonstration, clearly it is also better than that which leads to the impossible.

\[ \text{ἐπεὶ δ’ ἡ κατηγορικὴ τῆς στερητικῆς βελτίων, δήλον ὅτι καὶ τῆς εἰς τὸ ἀδύνατον ἀγούσης}. \] (APost. 1.26 87a1–2)

Direct demonstrations are better than demonstrations by reductio on the grounds that the former, but not the latter, proceed from premises which are ‘more familiar and prior’:

T2 Thus, if a demonstration which proceeds from what is more known and prior is superior, and if in both kinds of demonstration conviction proceeds from something’s not being the case, but in the one from something prior and in the other from something posterior, then privative demonstration will be better without qualification than demonstration leading to the impossible. Consequently, it is also clear that positive demonstration, which is better than privative demonstration, is better than that which leads to the impossible.

\[ \text{εἰ οὖν ἡ ἐκ γνωριμωτέρων καὶ προτέρων κρείττων, εἰσὶ δ’ ἀμφότεραι ἐκ τοῦ μὴ εἶναί τι πισταί, ἀλλ’ ἡ μὲν ἐκ προτέρου ἡ δ’ ἐξ ύστερου, βελτίων ἀπλῶς ἃν εἰ ἡ τῆς εἰς τὸ ἀδύνατον ἡ στερητικὴ ἀπόδειξις, ώστε καὶ ἡ ταύτης βελτίων ἡ κατηγορικὴ δῆλον ὅτι καὶ τῆς εἰς τὸ ἀδύνατον ἐστὶ βελτίων}. \] (APost. 1.26 87a25–30)

Aristotle distinguishes two kinds of priority: priority ‘in nature’ and priority ‘to us’. Likewise, he distinguishes between being more known ‘in nature’ and being more known ‘to us’ (APost. 1.2 71b33–72a5). The sense in which the premises of direct negative demonstrations are prior to the conclusion is priority in nature (φύσεi, 87a17).

Aristotle’s thesis in Posterior Analytics 1.26 has been widely influential:

T3 When geometers reason through the impossible, they are content merely to discover the property [of a given subject]. But when their reasoning proceeds through a principal demonstration, then, if such demonstration is partial, the cause is not yet clear, but if it is universal and applies to all like things, the ‘why’ is at once made evident.

\[ \text{ὅταν μὲν οὖν ὁ συλλογισμὸς ἢ δι’ ἀδύνατον τοῖς γεωμέτραις, ἀγαπῶσι τὸ σύμπτωμα μόνον εὑρεῖν, ὅταν δὲ διὰ προηγομένης ἀποδείξεως, τότε πάλιν, εἰ μὲν ἐπὶ μέρους αἱ ἀποδείξεις γίγνοντο, οὕπω δήλον τὸ αίτιον, εἰ δὲ καθ’ ὅλον καὶ ἐπὶ πάντων τῶν ὁμοίων, εὐθὺς καὶ τὸ διὰ τί γίγνεται καταφανές}. \] (Proclus in Eucl. I 202.19–25)
Following Proclus, Giuseppe Biancani (1615: 10, 12, and 20) denied that demonstrations by *reductio* proceed ‘from a cause’. In his *Lectiones mathematicae* from the 1660s, Isaac Barrow writes that ‘Aristotle teaches, and everyone grants, that reasoning of this sort [i.e., by *reductio ad impossibile*] does not at all furnish knowledge that is very perspicuous and pleasing to the mind’ (Barrow 1860: 377).

T4  Those demonstrations which show that a thing is such, not by its principles, but by some absurdity which would follow if it were not so, are very common in Euclid. It is clear, however, that while they may convince the mind, they do not enlighten it, which ought to be the chief result of knowledge; for our mind is not satisfied unless it knows not only that a thing is, but why it is, which cannot be learnt from a demonstration by reduction to the impossible. (Arnauld & Nicole, *Port-Royal Logic* IV 9.3)

T5  The third special rule of pure reason, if it is subjected to a discipline in regard to transcendental proofs, is that its proofs must never be apagogic but always ostensive. The direct or ostensive proof, in all kinds of cognition, is that which combines with the conviction of truth insight into the sources of the truth; the apagogic proof, by contrast, can produce certainty, but cannot enable us to comprehend the truth in its connection with the grounds of its possibility. Hence the latter is more of an emergency aid than a procedure which satisfies all the aims of reason. (Kant, *Critique of Pure Reason* A 789 / B 817)

Kant’s view of proofs by *reductio* was accepted, e.g., by Krug (1806: 595–7), Kiesewetter (1824: I 149 and II 492–3), and Reinhold (1827: 409–10). Bolzano argues that proofs by *reductio* do not exhibit ‘the objective ground’ of the *demonstrandum* (Bolzano 1837: IV §530, pp. 270–1).

Trendelenburg (1870: 440) writes: ‘Indirect proof, as Aristotle has already shown, possesses less scientific value than direct proof. … Indirect proof does not provide any insight into the inner grounds of the thing’.

At the same time, Aristotle’s argument in *Posterior Analytics* 1.26 has proved difficult to understand. Philoponus (in APost. 291.10–12) reports that ‘in the exposition of these things, all commentators together have attacked Aristotle (ἁπαξάπαντες οἱ ἐξηγηταὶ ἐπελάβοντο τοῦ Ἀριστοτέλους), saying that he expounds deduction through the impossible incorrectly’.

Zabarella (1608: 976a) writes: ‘I have considered this passage for a very long time, and have not found anything in other commentators in which I could quite acquiesce’.

Mignucci holds that ‘it is difficult to make sense of the confused argument that Aristotle advances’ in *Posterior Analytics* 1.26 (Mignucci 1975: 559). Smith (1982a: 131) maintains that ‘we cannot actually make c. 26 into a coherent account of per impossibile proof’ (1982a: 133). Detel (1993: 450) holds that Aristotle’s argument in *Posterior Analytics* 1.26 is ‘garbled’ and, ‘as a matter of fact, unsuccessful’.
2. Aristotle’s argument in *Posterior Analytics* 1.26

Aristotle’s syllogistic theory deals with four kinds of categorical proposition:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>AaB</td>
<td>A belongs to all B</td>
<td>(Every B is A)</td>
</tr>
<tr>
<td>AeB</td>
<td>A belongs to no B</td>
<td>(No B is A)</td>
</tr>
<tr>
<td>AiB</td>
<td>A belongs to some B</td>
<td>(Some B is A)</td>
</tr>
<tr>
<td>AoB</td>
<td>A does not belong to some B</td>
<td>(Not every B is A)</td>
</tr>
</tbody>
</table>

Aristotle explains the difference between direct negative demonstrations and demonstrations by *reductio ad impossibile* as follows:

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T6 We must understand what the difference between them is [i.e., between direct negative demonstration and demonstration by *reductio ad impossibile*]. Let A belong to no B and B to all C; it follows necessarily that A belongs to no C. If these premises are assumed, the privative demonstration that A does not belong to C will be ostensive. Demonstration leading to the impossible, on the other hand, proceeds as follows. If it is required to prove that A does not belong to B, we must assume that it does belong, and that B belongs to C; hence it follows that A belongs to C. But let it be known and agreed that this is impossible. Hence, A cannot belong to B. If, then, it is agreed that B belongs to C, it is impossible for A to belong to B.
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δεί δ’ εἰδέναι τίς ἡ διαφορὰ αὐτῶν. ἔστω δὴ τὸ Α μηδενὶ ὑπάρχον τῷ Β, τῷ δὲ Γ τὸ Β παντὶ· ἀνάγκη δὴ τῷ Γ μηδενὶ ὑπάρχειν τὸ Α. οὕτω μὲν ὃν μὴν ληφθέντων δεικτικὴ ἢ στερητικὴ ἢ ἀπὸ ἀπόδειξις ὅτι τὸ Α τῷ Γ οὐχ ὑπάρχει. ἡ δ’ εἰς τὸ ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύνατον ἀδύ

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1. AeC (premise)
2. BaC (premise)
3. AaB (assumption for *reductio*)
4. BaC (iterated from 2)
5. AaC (from 3, 4, by Barbara)
6. AoB (**reductio**: 1, 3–5)
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The inference from the premises in lines 1 and 2 to the conclusion in line 6 is an instance of *Felapton*: AeC, BaC, therefore AoB.

By nature, however, the proposition that A does not belong to B is prior to the proposition that A does not belong to C. For the things from which a conclusion derives are prior to the conclusion; and that A does not belong to C is the conclusion, whereas that A does not belong to B is that from which the conclusion derives. (APost. 1.26 87a17–20)

In T7 and T2, Aristotle states that direct demonstrations are better than those by reductio on the grounds that the former, unlike the latter, proceed from premises which are ‘prior in nature’ to the conclusion.

As Zabarella (1608: 975f–976a) points out, this seems to be in tension with Aristotle’s claim in Prior Analytics 2.14 that every conclusion deducible by means of reductio can be deduced from the same premises without reductio:

Everything concluded ostensively can also be proved through the impossible, and whatever is proved through the impossible can be concluded ostensively, through the same terms. (APr. 2.14 62b38–40; see also 63b12–21)

In view of this, Striker suggests that, in T7 and T2, Aristotle takes direct demonstrations to be better than indirect ones because of the way in which the conclusion is inferred from the premises (similarly, Themistius in Post. An. 37.3–7, Zabarella 1608: 976e–977d, Crivelli 2011: 170 n. 157):

As Aristotle himself shows in An. Pr. B 14, one can form a ‘genuine’ premise pair for the demonstrandum from the true premise of the syllogism [employed in the demonstration by reductio] and the negation of the impossible conclusion. However, Aristotle does not regard this further syllogism as a part of the reductio, but as a different proof of the same demonstrandum – in the reductio one infers the proposition to be proved ‘from a hypothesis’. … Aristotle’s argument for the superiority of ‘categorical’ proofs presumably relies on the fact that reductio proofs are not purely syllogistic. (Striker 1977: 318)

Barnes (1993: 188): ‘Aristotle’s argument does not turn on the nature of reductio as such’.

3. Aristotle’s two deductive systems:
When Aristotle asserts the equivalence of direct deduction and deduction by reductio in T8, he takes for granted the fourteen syllogistic moods established in Prior Analytics 1.4–6:

First figure: Barbara, Celarent, Darii, Ferio
Second figure: Cesare, Camestres, Festino, Baroco
Third figure: Darapti, Disamis, Datisi, Felapton, Ferison, Bocardo
In T8, Aristotle takes each of these fourteen moods to license a direct deduction. In this system, the rule of *reductio ad impossibile* is redundant: anything deductible from given premises by means of *reductio* can also be deduced from these premises without *reductio*.

**DEDUCTIVE SYSTEM D1 (Prior Analytics 1.2 and 1.4–6)**

1. **perfect moods:**
   - AaB, BaC, therefore AaC (Barbara)
   - AeB, BaC, therefore AeC (Celarent)
   - AaB, BiC, therefore AiC (Darii)
   - AeB, BiC, therefore AoC (Ferio)

2. **conversion rules:**
   - AeB, therefore BeA (e-conversion)
   - AiB, therefore BiA (i-conversion)
   - AaB, therefore BiA (a-conversion)

3. **Rule of *reductio ad impossibile***.

In D1, *reductio* is only needed to derive the moods Baroco and Bocardo. All other moods are derivable without *reductio*.

**DEDUCTIVE SYSTEM D2 (Prior Analytics 1.7):**

1. **perfect moods:**
   - AaB, BaC, therefore AaC (Barbara)
   - AeB, BaC, therefore AeC (Celarent)

2. **conversion rules:**
   - AeB, therefore BeA (e-conversion)

3. **Rule of *reductio ad impossibile***.

In D2, the only moods derivable by means of direct deductions without *reductio* are Cesare and Camestres. The rule of *reductio* is needed to derive all other moods: Darii, Ferio, Festino, Baroco, Darapti, Disamis, Datisi, Felapton, Ferison, and Bocardo.

In D2, Aristotle uses *reductio* to derive the moods Darii and Ferio from Celarent (1.7 29b1–19; see Alexander *In Pr. An.* 113.5–114.30, Weidemann 2004: 73–5, Barnes 2007: 364–6).

**4. Priority in nature for a-propositions**

In T2, Aristotle claims that direct negative demonstration is better than demonstration by *reductio* because ‘the one proceeds from something prior and the other from something posterior’.

Given Aristotle’s definition of demonstration in *Posterior Analytics* 1.2, the claim that the premises of demonstrations by *reductio* are not prior in nature to the conclusion is contradictory. Likewise, the claim that every direct demonstration proceeds from premises which are prior in nature to the conclusion is trivial.

Suggestion: Aristotle’s claim in T2 is that every direct deduction, but not every deduction by *reductio*, in which the premises are scientific propositions proceeds from premises which are prior in nature to the conclusion.
In order to verify the latter claim, we need an account of what it is for a scientific proposition to be prior in nature to another scientific proposition. Such an account can be supplied by means of the chains (συστοιχίαι) of terms connected by immediate a-predications considered by Aristotle in *Posterior Analytics* 1.19–25:

T10 Let C be such that it itself does not further belong to any other thing, and that B belongs to it directly, with nothing else between them. Again, let E belong to F in the same way, and F to B.

\[\text{ἔστω δὴ τὸ Γ τοιοῦτον δ ἀυτὸ μὲν μηκέτι ὑπάρχει ἄλλῳ, τούτῳ δὲ τὸ Β πρώτῳ, καὶ οὐκ ἔστιν ἄλλο μεταξύ. καὶ πάλιν τὸ E τῷ Z ὑσαύτως, καὶ τούτῳ τῷ Β. (APost. 1.19 81b30–2)\]

The chain of immediate a-predications described in T10 is:

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E --- F --- B --- C
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T11 Let one demonstration show that A belongs to E through the middle terms B, C, and D, and let the other show that A belongs to E through F and G. Thus the proposition that A belongs to D and the one that A belongs to E are on a par. But that A belongs to D is prior to and more known than that A belongs to E; for the latter is demonstrated through the former.

\[\text{ἔστω ἡ μὲν διὰ μέσων ἀπόδειξις τῶν Β Γ Δ ὅτι τὸ Α τῷ Ε ὑπάρχει, ἡ δὲ διὰ τῶν Ζ Η ὅτι τὸ Α τῷ Ε. ὁμοίως δὴ ἔχει τὸ ὅτι τὸ Α τῷ Δ ὑπάρχει καὶ τὸ Α τῷ Ε. τὸ δ᾽ ὅτι τὸ Α τῷ Δ πρότερον καὶ γνωριμώτερον ἢ ὅτι τὸ Α τῷ Ε· διὰ γὰρ τούτου ἐκεῖνο ἀποδείκνυται. (APost. 1.25 86a39–86b5)\]

**Priority in nature for a-propositions:** an a-proposition AaB is prior in nature to an a-proposition CaD just in case there is an a-path from A to B which is a proper part of some a-path from C to D.

Now, the relation of priority in nature is asymmetric: if one proposition is prior in nature to another, the latter is not prior in nature to the former (APost. 1.3 72b27–8).

The relation of priority in nature just defined is not asymmetric if there are cycles of a-predications, such as AaB, BaC, CaA. In this cycle, BaC is prior in nature to AaC, and vice versa.
The a-paths (or, chains of predications) considered by Aristotle in book 1 of the *Posterior Analytics* do not contain any cycles. This is because these a-paths do not allow for counterpredication:

T12  
[i] If this is not a quality of that and that of this – a quality of a quality –, then it is impossible for one thing to be counterpredicated of another in this way. It is possible to make a true statement, but it is not possible to counterpredicate truly. [ii] For either it will be predicated as substance, i.e. being either the genus or the differentia of what is predicated. . . . Surely they will not be counterpredicated of one another as genera; for then something would be just what is some of itself. [iii] Nor will anything be counterpredicated of a quality or the other kinds of thing – unless it is predicated accidentally; for all these things are accidents, and they are predicated of substances.


Consequently, T12 excludes cycles of predication of the form: $A_1$ is predicated of $A_2$, $A_2$ of $A_3$, ..., $A_{n-1}$ of $A_n$, and $A_n$ of $A_1$ (Ross 1949: 578–9, Lear 1979: 214). For, given that predication is transitive, any two terms in such a cycle are predicated of each other.

Given that a-predication is acyclic, the relation of priority in nature between a-propositions defined above is asymmetric. Thus, any given science $S$ determines an acyclic a-structure, in which the relation of immediate a-predication is represented by an arrow, $\rightarrow$:

**DEFINITION 1**: An *a-structure* consists of a set of terms equipped with a binary acyclic relation, $\rightarrow$. Thus, there are no terms $A_1, A_2, ..., A_n$ ($n \geq 1$) such that $A_i \rightarrow A_{i+1}$ ($1 \leq i \leq n-1$) and $A_n \rightarrow A_1$.

An a-path is a sequence of terms connected by immediate universal affirmations:
DEFINITION 2: Given an a-structure, an a-path is a sequence of terms $A_1, A_2, \ldots, A_n$ ($n > 1$) such that $A_i \rightarrow A_{i+1}$ for all $1 \leq i \leq n-1$.

A-paths can be represented by acyclic directed graphs:

![Diagram of an a-path]

5. Priority in nature for e-propositions

In addition to immediate a-propositions, Aristotle also considers immediate e-propositions:

T13 Just as it is possible for $A$ to belong to $B$ atomically, so it is also possible for it atomically not to belong. By atomically belonging or not belonging I mean that there is no middle term for them; for, in this case, they no longer belong or do not belong by virtue of something else.

\[ \text{естественнее} \tauο \, Α \, τό \, Β \, \text{ένεδέχετο} \, \text{άτομως, ούτω καὶ μή υπάρχειν \, \text{έγχωρεῖ.}} \, \text{λέγω \, δὲ \, το \, \text{άτομως \, υπάρχειν} \, ή \, μή \, υπάρχειν} \, \tauό \, \text{μή \, \text{εἴναι} \, \text{αὐτῶν} \, \text{μέσον· ούτω γάρ οὐκέτι \, ἔσται kατ' \, ἄλλο} \, \tauό \, \text{υπάρχειν} \, \, ή \, μή \, υπάρχειν.} \quad (\text{APost. 1.15 79a33–6}) \]

Aristotle considers complex negative demonstrations in which a theorem is demonstrated by successive applications of Celarent, Cesare, Camestres, and Barbara. In these demonstrations, the conclusion corresponds to complex e-paths such as the following:

![Diagram of an e-path]

In this e-path, the terms $B_3$ and $B_4$ are disjoint in the sense that there is no a-path from the one to the other and no two a-paths starting from these two terms meet in a third term:

DEFINITION 3: In any a-structure, two terms $A$ and $B$ are disjoint just in case

(i) $A$ is distinct from $B$,
(ii) there is no a-path from $A$ to $B$ or vice versa, and
(iii) there is no term $C$ in the a-structure such that there is an a-path from $A$ to $C$ and an a-path from $B$ to $C$.

Hence, an e-path can be characterized as a sequence of terms $A_1, A_2, \ldots, A_n$ in which one term, $A_n$, is disjoint from its successor, $A_{n+1}$. If $A_{n+1}$ is not the endpoint of the e-path, $A_n$, then there is an a-path from the former to the latter. Likewise, if $A_i$ is not the starting point of the e-path, $A_1$, then there is an a-path from the former to the latter:
DEFINITION 4: Given an a-structure, an e-path is a sequence of terms $A_1, A_2, \ldots, A_n$ ($n > 1$) such that for some $A_i (1 \leq i < n)$:

1. $A_i$ is disjoint from $A_{i+1}$,
2. either $i = 1$ or $A_n, A_{i-1}, \ldots, A_1$ is an a-path, and
3. either $i+1 = n$ or $A_{i+1}, A_{i+2}, \ldots, A_n$ is an a-path.

DEFINITION 5: A subpath of an a-path $A_1, A_2, \ldots, A_n$ is any a-path $A_i, A_{i+1}, \ldots, A_j$ ($1 \leq i \leq j \leq n$) such that either $1 < i$ or $j < k$ (or both).

The subpaths of an e-path are all e-paths that are proper parts of it, and all subpaths of the two a-paths that may be contained in it:

DEFINITION 6: Let $A_1, A_2, \ldots, A_n$ be an e-path in which $A_i$ and $A_{i+1}$ are disjoint ($1 \leq i < n$). A subpath of this e-path is:

1. any subpath of the a-path $A_i, A_{i-1}, \ldots, A_1$ (if $i \neq 1$),
2. any subpath of the a-path $A_{i+1}, A_{i+2}, \ldots, A_n$ (if $i+1 \neq n$), and
3. any e-path of the form $A_k, \ldots, A_i, A_{i+1}, \ldots, A_m$ ($1 \leq k \leq i$ and $i+1 \leq m \leq n$) such that either $1 < k$ or $m < n$ (or both).

DEFINITION 7: For any a-structure and any terms A, B in this a-structure:

- $AaB$ is satisfied in the a-structure iff there is an a-path from A to B
- $AeB$ is satisfied in the a-structure iff there is an e-path from A to B
- $AiB$ is satisfied in the a-structure iff there is no e-path from A to B
- $AoB$ is satisfied in the a-structure iff there is no a-path from A to B

THEOREM 1: In any a-structure, if there is an a-path from A to B, then there is no e-path from A to B.

DEFINITION 8: In any a-structure, a path is either an a-path or an e-path.

Priority in nature for any scientific propositions of the form $AaB$, $AeB$, $AiB$, and $AoB$ can now be defined as follows:

DEFINITION 9: For any propositions $AxB$ and $CyD$ that are satisfied in an a-structure (where ‘x’ and ‘y’ are placeholders for ‘a’, ‘e’, ‘i’, and ‘o’): $AxB$ is prior in nature to $CyD$ just in case some path from A to B is a subpath of some path from C to D.

This relation of priority is asymmetric:

THEOREM 2: The relation of priority in nature introduced in Definition 9 is transitive and irreflexive.
6. Accounting for Aristotle’s thesis in *Posterior Analytics* 1.26

**Theorem 3:** For any instance of Barbara, Celarent, Cesare, and Camestres: if the premises are satisfied in an a-structure, then each premise is prior in nature to the conclusion.

The theorem holds because, in any a-structure, reasoning by Barbara amounts to extending a-paths, and reasoning by Celarent, Cesare, and Camestres amounts to extending e-paths.

This is not the case for deductions by *reductio* in D2, in which indirect deductions from true scientific premises may proceed from posterior to prior propositions:

**Theorem 4:** There are instances of Darii, Ferio, Festino, Darapti, Disamis, Datisi, Felapton, Ferison, and Bocardo in which both premises are satisfied in an a-structure and the conclusion is prior in nature to one of the premises.

For the case of Felapton (AeC, BaC, therefore AoB), this can be seen from the same arrangement of terms used by Aristotle in T6:

![Diagram](image)

There are analogous results in contemporary theories of grounding. Consider a deductive system for the language of propositional logic using the connectives of conjunction ($\land$) and negation ($\neg$). The deductive system includes four direct rules of inference:

- $\varphi, \psi$, therefore $\varphi \land \psi$
- $\neg \varphi$, therefore $\neg (\varphi \land \psi)$
- $\neg \psi$, therefore $\neg (\varphi \land \psi)$
- $\varphi$, therefore $\neg \neg \varphi$

In addition, the deductive system contains the following rule of *reductio ad impossibile*:

\[
\frac{\Gamma, \neg \varphi, \therefore \psi \land \neg \psi}{\Gamma, \therefore \varphi}
\]

The four direct rules are special in that, when applied to true propositions, the conclusion is also grounded in the premises. The relation of ground is usually taken to obey the following laws corresponding to the four direct rules of inference (Fine 2012a: 58 and 62–3, Correia 2014: 33–6, 2017: 530; similarly, Schnieder 2011: 449, Korbmacher 2018: 170):

- If $\varphi$ and $\psi$, then $\varphi, \psi$ ground $\varphi \land \psi$
- If $\neg \varphi$, then $\neg \varphi$ grounds $\neg (\varphi \land \psi)$
- If $\neg \psi$, then $\neg \psi$ grounds $\neg (\varphi \land \psi)$
- If $\varphi$, then $\varphi$ grounds $\neg \neg \varphi$
Thus, when the four direct rules of inference are applied to true propositions, the premises ground the conclusion. This is not true for deductions that employ *reductio*. For example, the above rule of *reductio* licenses the following derived rules in the system:

\[
\begin{align*}
\varphi \land \psi, & \quad \text{therefore } \varphi \\
\varphi \land \psi, & \quad \text{therefore } \psi \\
\neg \neg \varphi, & \quad \text{therefore } \varphi
\end{align*}
\]

Aristotle’s relation of priority in nature is comparable to the relation of strict partial ground described by Fine (2012a: 71–3, 2012b: 7–9). On Fine’s account, if a truth \( \varphi \) is a strict partial ground of a truth \( \psi \), then there are truths \( \varphi_1, \ldots, \varphi_n \) such that, for any facts \( f, f_1, \ldots, f_n \): if \( f \) verifies \( \varphi \) and \( f_i \) verifies \( \varphi_i \) for all \( 1 \leq i \leq n \), then the fusion of the facts \( f, f_1, \ldots, f_n \) is a fact that verifies \( \psi \).

Thus, if \( \varphi \) is a strict partial ground of \( \psi \), then every verifier of \( \varphi \) is part of a verifier of \( \psi \).

Now, let us say that a scientific proposition \( AxB \) is *verified* by any path from \( A \) to \( B \) in an a-structure.

Then the characterization of priority in nature given in Definition 9 entails that, if a scientific proposition \( AxB \) is prior in nature to a scientific proposition \( CyD \), then there are scientific propositions \( \varphi_1, \ldots, \varphi_n \) such that, for any paths \( P, P_1, \ldots, P_n \): if \( P \) verifies \( AxB \) and \( P_i \) verifies \( \varphi_i \) for all \( 1 \leq i \leq n \), then there exists a path which is the concatenation of the paths \( P, P_1, \ldots, P_n \) and which verifies \( CyD \).

Thus, if \( AxB \) is prior in nature to \( CyD \), then every verifier of \( AxB \) is part of a verifier of \( CyD \).

### 7. The relation between the *Prior Analytics* and the first book of the *Posterior Analytics*

In the *Posterior Analytics*, Aristotle focuses on the four universal moods Barbara, Celarent, Cesare, and Camestres. Smith (1982b: 327–35, 1982a: 114–35) takes this to show that the *Posterior Analytics* was largely composed before Aristotle developed the mature syllogistic theory presented in the *Prior Analytics*.

Theorems 3 and 4 suggest another explanation of Aristotle’s focus in the *Posterior Analytics* on the four universal moods. These are the only moods in which the premises are guaranteed to be prior in nature to the conclusion when applied to scientific propositions.

There is no reason to maintain that *Posterior Analytics* 1.26 was written before *Prior Analytics* 2.11–14 because otherwise one ‘must suppose that Aristotle, after having acquired the sort of understanding of *reductio ad impossibile* reflected in An. Pr. 2.11–14, … somehow produced the very unsatisfactory treatment of this subject in An. Post. 1.26’ (Smith 1982a: 134).
Solmsen (1929: 53–7 and 81–122) has argued that the account of demonstration given by Aristotle in the first book of the Posterior Analytics is based on chains of terms arranged in order of increasing generality. Solmsen (1941: 410–20) refers to these chains as *Eidosketten*.

Smith (1982a: 122) argues that Aristotle’s ‘theory of demonstration is the theory of the structure of a system of terms arranged in ordered ‘chains’ (συστοιχίαι)’.

I have argued that, for Aristotle, any given science determines an a-structure. Given that these structures are acyclic, they give rise to a well-defined relation of priority in nature among scientific propositions. It is this relation of priority in nature that lies at the heart of Aristotle’s argument in Posterior Analytics 1.26.

References:

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